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# Lowest weight modules of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ of minimal Gelfand–Kirillov dimension

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## Abstract

This paper is to classify genuine irreducible lowest weight modules of  $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ , of Gelfand–Kirillov dimension  $n$ .

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**Keywords:** Lowest weight module; Gelfand–Kirillov dimension; Oscillator representation; Metaplectic group

## 1. Introduction

Fix a positive integer  $n$ . The metaplectic group  $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$  is the unique non-split double covering of the real symplectic group  $\mathrm{Sp}_{2n}(\mathbb{R})$ , and we have an exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow G \xrightarrow{\eta} \mathrm{Sp}_{2n}(\mathbb{R}) \rightarrow 1$$

of Lie groups. Write  $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ , which is the complexified Lie algebra of  $G$ . The unitary group  $U(n)$  is a maximal compact subgroup of  $\mathrm{Sp}_{2n}(\mathbb{R})$ . Write  $K = \eta^{-1}(U(n))$ , which is a maximal compact subgroup of  $G$ . By abuse of notation, we do not distinguish representations of  $G$  and the underlying  $(\mathfrak{g}, K)$  modules.

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The size of a representation of  $G$  is measured by its Gelfand–Kirillov dimension. We recall the definition briefly here. For more details, see [2,10]. Let “ $\mathcal{U}$ ” stand for universal enveloping algebras. Write  $\mathcal{U}_k(\mathfrak{g})$  for the subspace of  $\mathcal{U}(\mathfrak{g})$  spanned by products of at most  $k$  elements of  $\mathfrak{g}$ . Let  $M$  be a non-zero finitely generated admissible  $(\mathfrak{g}, K)$  module. Take a finite-dimensional subspace  $M_0$  of  $M$  which generates  $M$  as a  $\mathcal{U}(\mathfrak{g})$  module. By the theory of Hilbert polynomials, there are a non-negative integer  $d_M$  and a positive integer  $B_M$  such that

$$\dim \mathcal{U}_k(\mathfrak{g})M_0 = \frac{B_M}{d_M!} k^{d_M} + O(k^{d_M-1}), \quad \text{as } k \rightarrow +\infty. \quad (1)$$

The integer  $d_M$  is called the Gelfand–Kirillov dimension of  $M$ , and  $B_M$  is called the Bernstein degree. They are independent of the choice of  $M_0$ .

Small representations are of great interest in representation theory. See [11] for a general introduction. The smallest possible value of  $d_M$  is 0, and  $d_M = 0$  if and only if  $M$  is finite-dimensional. In this case, irreducible representations of  $G$  are classified by the classical highest weight theory, and all the representations descend to representations of  $\mathrm{Sp}_{2n}(\mathbb{R})$ . By the theory of associated varieties [9,11,13], one knows that the next smallest possible value of  $d_M$  is  $n$ . There is a famous example of representation of  $G$  with Gelfand–Kirillov dimension  $n$ , namely, the oscillator representation. Actually, there are two oscillator representations of  $G$ , which are contragredient to each other. We only consider one of them. So the oscillator representation  $\omega$  under consideration is the direct sum of two irreducible unitary representations, which satisfy

$$\begin{cases} \text{(a)} & \text{they are genuine lowest weight modules, and} \\ \text{(b)} & \text{they have Gelfand–Kirillov dimension } n. \end{cases} \quad (2)$$

Recall that a representation of  $G$  is said to be genuine if the kernel  $\{\pm 1\}$  acts via the unique non-trivial character. The usual terminology “lowest weight module” will be explained later.

We are interested in finding all representations which are “close” to oscillator representations. More precisely, the goal of this paper is a classification of irreducible  $(\mathfrak{g}, K)$  modules, unitary or not, with the two properties of (2). The original motivation of the classification comes from calculations of nilpotent cohomologies of oscillator representations, which generalize J. Adams’ results in [1], and are expected to be used in explicit theta correspondences. Unitary lowest (or highest) weight modules are classified and studied extensively in literature. See [3–5,7–9] for example. In particular, the Gelfand–Kirillov dimensions of irreducible unitary lowest weight modules of  $G$  are calculated in [5] and [9]. Both papers use theta correspondences of compact dual pairs, and their methods do not apply to non-unitary representations.

Let us introduce more notation. We have a diagram

$$\begin{array}{ccccccccc} Z & \subset & C & \subset & T & \subset & K & \subset & G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow_{\eta} \\ \{\pm 1\} & \subset & \mathrm{U}(1) & \subset & \mathrm{U}(1)^n & \subset & \mathrm{U}(n) & \subset & \mathrm{Sp}_{2n}(\mathbb{R}) \end{array}$$

of groups, where  $\mathrm{U}(1) \subset \mathrm{U}(1)^n$  is the diagonal embedding,

$$T = \eta^{-1}(\mathrm{U}(1)^n), \quad C = \eta^{-1}(\mathrm{U}(1)), \quad \text{and} \quad Z = \eta^{-1}(\{\pm 1\}).$$

Then  $T$  is a maximal torus of  $K$ ,  $C$  is the center of  $K$ , and  $Z$  is the center of  $G$ . The character group of  $U(1)^n$  is identified with  $\mathbb{Z}^n$ , and that of  $T$  is identified with

$$\Lambda \doteq \mathbb{Z}^n \cup \left( \frac{1}{2} + \mathbb{Z} \right)^n \subset \mathbb{R}^n.$$

The characters of  $U(1)$  are identified with  $\mathbb{Z}$ , and those of  $C$  are identified with

$$\begin{cases} \frac{1}{2}\mathbb{Z}, & \text{if } n \text{ is odd,} \\ \mathbb{Z} \times (\Lambda/\mathbb{Z}^n), & \text{if } n \text{ is even.} \end{cases}$$

By restricting from  $C$ , the characters of  $Z$  are identified with

$$\begin{cases} \frac{1}{2}\mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z}) \times (\Lambda/\mathbb{Z}^n), & \text{if } n \text{ is even.} \end{cases}$$

Call the corresponding character

$$\begin{cases} \chi_s, & s \in \frac{1}{2}\mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is odd,} \\ \chi_{s,\epsilon}, & (s,\epsilon) \in (\mathbb{Z}/2\mathbb{Z}) \times (\Lambda/\mathbb{Z}^n), & \text{if } n \text{ is even.} \end{cases}$$

The root system of  $\mathfrak{g}$  (relative to  $T$ ) is

$$\Delta \doteq \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq n\},$$

and the compact roots are

$$\Delta_c \doteq \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n, i \neq j\},$$

where  $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$  is the standard basis of  $\mathbb{R}^n$ . Fix a positive system

$$\Delta^+ \doteq \{-\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\},$$

and write the negative roots

$$\Delta^- = -\Delta^+.$$

The half sum of the positive roots is

$$\rho = (1, 2, \dots, n).$$

The Lie algebra  $\mathfrak{g}$  has vector space decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- \quad \text{and} \quad \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

where  $\mathfrak{k}$  and  $\mathfrak{t}$  are the complexified Lie algebras of  $K$  and  $T$ , respectively,  $\mathfrak{p}^\pm$  is the  $K$  stable abelian subspace corresponding to the roots

$$\pm(\{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\}),$$

and  $\mathfrak{n}^\pm$  is the  $T$  stable nilpotent Lie subalgebra corresponding to the roots  $\Delta^\pm$ .

A  $(\mathfrak{g}, K)$  module  $M$  is called a lowest weight module if it is generated by  $M^{\mathfrak{n}^-}$ . Here and henceforth, a superscript Lie algebra stands for the Lie algebra invariants. A  $T$  eigenvector in  $M^{\mathfrak{n}^-}$  is called a lowest weight vector of  $M$ , and the corresponding eigenvalue, which is a character of  $T$ , is called a lowest weight.

Write

$$\Lambda_{\text{low}} \doteq \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda \mid \lambda_1 > \lambda_2 > \dots > \lambda_n\}.$$

Let

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_{\text{low}}.$$

Denote by  $E_{\lambda+\rho}$  an irreducible finite-dimensional representation of  $K$  of lowest weight  $\lambda + \rho$ . The generalized Verma module, which is a lowest weight  $(\mathfrak{g}, K)$  module, is defined to be

$$M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}+\mathfrak{p}^-)} E_{\lambda+\rho},$$

with  $\mathcal{U}(\mathfrak{g})$  acts by left multiplication, and  $K$  acts by

$$k(a \otimes v) = \text{Ad}_k(a) \otimes av, \quad k \in K, \quad a \in \mathcal{U}(\mathfrak{g}), \quad v \in E_{\lambda+\rho},$$

where “Ad” stands for the adjoint representation. The irreducible lowest weight module  $L_\lambda$  is by definition the unique irreducible quotient of  $M_\lambda$ . Both  $M_\lambda$  and  $L_\lambda$  have infinitesimal character  $\lambda$  and have central character

$$\begin{cases} \chi_{s(\lambda)}, & \text{if } n \text{ is odd,} \\ \chi_{s(\lambda), \lambda + \mathbb{Z}^n}, & \text{if } n \text{ is even,} \end{cases}$$

where

$$s(\lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n + 1 + 2 + \dots + n + 2\mathbb{Z}.$$

Let us remark here that the Gelfand–Kirillov dimension of  $M_\lambda$  is

$$\dim(\mathfrak{p}^+) = \frac{n^2 + n}{2},$$

which is much bigger than  $n$  for  $n$  large.

Irreducible lowest weight modules are parametrized by  $\Lambda_{\text{low}}$ , i.e., we have a bijection

$$\begin{aligned} \Lambda_{\text{low}} &\leftrightarrow \{\text{isomorphic class of irreducible lowest weight } (\mathfrak{g}, K) \text{ modules}\}, \\ \lambda &\mapsto L_\lambda. \end{aligned}$$

Set

$$\Lambda_{\text{gnl}} \doteq \Lambda_{\text{low}} \cap \left(\frac{1}{2} + \mathbb{Z}\right)^n,$$

which parametrizes irreducible genuine lowest weight modules. Notice that the oscillator representation

$$\omega \cong L_{(-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots, -\frac{2n-1}{2})} \oplus L_{(\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots, -\frac{2n-1}{2})}.$$

The main theorem of the paper is

**Theorem 1.1.** *Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_{\text{gnl}}$ . Then  $L_\lambda$  has Gelfand–Kirillov dimension  $n$  if and only if  $n = 1$ , or*

$$n \geq 2 \quad \text{and} \quad -|\lambda_1| > \lambda_2.$$

To shorten terminology, we call a  $(\mathfrak{g}, K)$  module oscillator-like if it is finitely generated, admissible, genuine, and every irreducible subquotient of it is a lowest weight module of Gelfand–Kirillov dimension  $n$ . It is clear that

- (a) every subquotient of an oscillator-like representation is still oscillator-like, and
- (b) the tensor product of a finite-dimensional representation with an oscillator-like representation is still oscillator-like.

Now irreducible oscillator-like representations are parametrized by

$$\Lambda_{\text{osc}} \doteq \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \left( \frac{1}{2} + \mathbb{Z} \right)^n \mid -|\lambda_1| > \lambda_2 > \dots > \lambda_n \right\}.$$

Among them, when  $n \geq 2$ , only the two summands of  $\omega$  are unitary. We draw two direct consequences of the above theorem. They reflect the fact that oscillator-like representations are “close” to finite-dimensional ones.

**Corollary 1.2.** *Up to isomorphism, an irreducible oscillator-like  $(\mathfrak{g}, K)$  module is determined by its infinitesimal character together with its central character.*

**Corollary 1.3.** *Every oscillator-like  $(\mathfrak{g}, K)$  module is completely reducible. In particular, the tensor product of the oscillator representation with an arbitrary finite-dimensional representation is completely reducible.*

The other result we get in the proof of the main theorem is

**Corollary 1.4.** *Let  $\lambda \in \Lambda_{\text{gnl}}$ . Then  $L_\lambda$  has Gelfand–Kirillov dimension  $n$  if and only if it occurs in*

$$\omega \otimes F$$

*for some finite-dimensional representation  $F$  of  $G$ .*

## 2. The case of $n = 1$

Let us consider the case of  $n = 1$  first.

**Lemma 2.1.** *If  $n = 1$  and  $\lambda \in \Lambda_{\text{gnl}} = \frac{1}{2} + \mathbb{Z}$ , then  $M_\lambda$  is irreducible. Consequently,  $L_\lambda$  has Gelfand–Kirillov dimension 1.*

**Proof.** This is easy and well known as  $\lambda$  is not a negative integer.  $\square$

**Lemma 2.2.** *Assume  $n = 1$ , and let  $\lambda, \lambda' \in \Lambda_{\text{gnl}} = \frac{1}{2} + \mathbb{Z}$ . If  $L_\lambda$  and  $L_{\lambda'}$  have the same infinitesimal character and the same central character, then  $\lambda = \lambda'$ .*

**Proof.** The conditions imply

$$|\lambda| = |\lambda'| \quad \text{and} \quad \lambda - \lambda' \in 2\mathbb{Z},$$

which force  $\lambda = \lambda'$  as  $\lambda$  is not an integer.  $\square$

**Lemma 2.3.** *Assume  $n = 1$ , and let  $\lambda, \lambda' \in \Lambda_{\text{gnl}} = \frac{1}{2} + \mathbb{Z}$ . Then every exact sequence*

$$0 \rightarrow L_\lambda \rightarrow N \rightarrow L_{\lambda'} \rightarrow 0$$

*of  $(\mathfrak{g}, K)$  modules splits.*

**Proof.** If  $L_\lambda$  and  $L_{\lambda'}$  have different infinitesimal characters or different central characters, then the exact sequence clearly splits. Otherwise Lemma 2.2 implies that  $\lambda = \lambda'$ . Then the weight  $\lambda + \rho$  has multiplicity 2 in  $N$ . Take  $v \in N \setminus L_\lambda$ , of weight  $\lambda + \rho$ . Then  $v$  generate a submodule which is isomorphic to  $L_{\lambda'}$  via the projection  $N \rightarrow L_{\lambda'}$ .  $\square$

We call a  $T$  module  $N$  bounded below if there is a real number  $B_0$  such that

$$a_i \geq B_0, \quad i = 1, 2, \dots, n,$$

for every weight  $(a_1, a_2, \dots, a_n) \in \Lambda$  of  $N$ . Notice that both  $M_\lambda$  and  $L_\lambda$  are bounded below for all  $\lambda \in \Lambda_{\text{low}}$ .

**Lemma 2.4.** *Assume  $n = 1$ . If  $N$  is a bounded below genuine  $(\mathfrak{g}, K)$  module, then  $N$  is completely reducible.*

**Proof.** Without loss of generality, we assume that  $N$  is finitely generated. Then  $N$  is automatically of finite length since it is bounded below. Let  $S$  be the socle of  $N$ . Assume  $S \neq N$  by contradiction. Then there exist a submodule  $N'$  of  $N$ , a half integer  $\lambda' \in \frac{1}{2} + \mathbb{Z}$ , and a non-split exact sequence

$$0 \rightarrow S \rightarrow N' \rightarrow L_{\lambda'} \rightarrow 0.$$

Therefore  $\text{Ext}_{(\mathfrak{g}, K)}^1(L_{\lambda'}, S) \neq 0$ , and consequently,  $\text{Ext}_{(\mathfrak{g}, K)}^1(L_{\lambda'}, L_\lambda) \neq 0$  for some irreducible summand  $L_\lambda$  of  $S$ . This contradicts Lemma 2.3.  $\square$

**Lemma 2.5.** Assume  $n = 1$ . If  $N$  is as in Lemma 2.4, then the action map  $\mathcal{U}(\mathfrak{p}^+) \otimes N^{\mathfrak{p}^-} \rightarrow N$  is bijective.

**Proof.** This follows easily from Lemma 2.1 and Lemma 2.4.  $\square$

Now we return to the general case. Decompose

$$\mathfrak{p}^+ = \mathfrak{p}_l^+ \oplus \mathfrak{p}_s^+,$$

where  $\mathfrak{p}_l^+$  is the  $T$  stable subspace corresponding to the long roots

$$\{2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_n\},$$

and  $\mathfrak{p}_s^+$  is the  $T$  stable subspace corresponding to the short roots

$$\{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\}.$$

Similarly, we have a decomposition

$$\mathfrak{p}^- = \mathfrak{p}_l^- \oplus \mathfrak{p}_s^-.$$

Write

$$\mathfrak{g}_l = \mathfrak{t} \oplus \mathfrak{p}_l^+ \oplus \mathfrak{p}_l^-,$$

which is isomorphic to  $\mathfrak{sl}_2(\mathbb{C})^n$  as a Lie algebra. Set  $\rho_l = (1, 1, \dots, 1)$ . For all  $\lambda \in (\mathbb{Z} + \frac{1}{2})^n$ , we define the Verma module

$$M_{l,\lambda} := \mathcal{U}(\mathfrak{g}_l) \otimes_{\mathcal{U}(\mathfrak{t} + \mathfrak{p}_l^-)} \mathbb{C}_{\lambda + \rho_l},$$

which is a genuine  $(\mathfrak{g}_l, T)$  module, where  $\mathbb{C}_{\lambda + \rho_l}$  is the one-dimensional  $T$  module of weight  $\lambda + \rho_l$ . The terminologies “Gelfand–Kirillov dimension,” “genuine” and “lowest weight module” also apply to  $(\mathfrak{g}_l, T)$  modules. Everything works exactly as in the case of  $n = 1$ , and we summarize as

**Lemma 2.6.** Let  $\lambda, \lambda' \in (\frac{1}{2} + \mathbb{Z})^n$ .

- (a) The Verma module  $M_{l,\lambda}$  is irreducible, and has Gelfand–Kirillov dimension  $n$ .
- (b) If  $M_{l,\lambda}$  and  $M_{l,\lambda'}$  have the same infinitesimal character and the same central character, then  $\lambda = \lambda'$ .
- (c) Every exact sequence

$$0 \rightarrow M_{l,\lambda} \rightarrow N \rightarrow M_{l,\lambda'} \rightarrow 0$$

of  $(\mathfrak{g}_l, T)$  modules splits.

- (d) If  $N$  is a bounded below genuine  $(\mathfrak{g}_l, T)$  module, then  $N$  is completely reducible.
- (e) If  $N$  is as in part (d), then the action map  $\mathcal{U}(\mathfrak{p}_l^+) \otimes N^{\mathfrak{p}_l^-} \rightarrow N$  is bijective.

### 3. Lowest weight modules of minimal Gelfand–Kirillov dimension

Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_{\text{gnl}}$  throughout this section.

The main purpose of this section is to prove

**Proposition 3.1.** *Assume  $n \geq 2$ . If  $L_\lambda$  has Gelfand–Kirillov dimension  $n$ , then  $\lambda \in \Lambda_{\text{osc}}$ , i.e.,*

$$-|\lambda_1| > \lambda_2.$$

Let  $\mathbb{Z}((\Lambda))$  be the group of integer valued functions on  $\Lambda$ . Its elements are formally written as

$$\sum_{\tau \in \Lambda} a_\tau e^\tau,$$

with all  $a_\tau \in \mathbb{Z}$ . Let  $\Lambda^+ \subset \Lambda$  be the non-negative integral combinations of elements of  $\Delta^+$ . Write  $\mathcal{R}$  for the functions in  $\mathbb{Z}((\Lambda))$  whose supports are contained in finite unions of translations of  $\Lambda^+$ . Use the convolution as multiplication,  $\mathcal{R}$  is a commutative integral domain. For every admissible representation  $N$  of  $T$ , write

$$\text{ch}(N) = \sum_{\tau \in \Lambda} a_\tau(N) e^\tau \in \mathbb{Z}((\Lambda)),$$

and call it the formal character of  $N$ , where  $a_\tau(N)$  is the multiplicity of  $\tau$  in  $N$ . See [6] for more details. Notice that  $\text{ch}(N) \in \mathcal{R}$  when  $N$  is a finitely generated admissible  $(\mathfrak{g}, K)$  module which is bounded below.

It follows from part (e) of Lemma 2.6 that

**Lemma 3.2.** *Assume  $N$  is a finitely generated admissible genuine  $(\mathfrak{g}, K)$  module which is bounded below. Then*

$$\text{ch}(N) = \text{ch}(\mathcal{U}(\mathfrak{p}_l^+)) \text{ch}(N^{\mathfrak{p}_l^-}). \quad (3)$$

**Lemma 3.3.** *Assume  $N$  is as in Lemma 3.2. If  $N$  has Gelfand–Kirillov dimension  $n$ , then*

$$\dim(N^{\mathfrak{p}_l^-}) < \infty.$$

**Proof.** Assume  $N$  has Gelfand–Kirillov dimension  $n$ , and has Bernstein degree  $B$ . Let  $N'$  be a finite-dimensional subspace of  $N^{\mathfrak{p}_l^-}$ . Take a finite-dimensional subspace  $N_0$  of  $N$  which contains  $N'$  and generates  $N$  as a  $\mathcal{U}(\mathfrak{g})$  module. Then

$$\begin{aligned} \dim(\mathcal{U}_k(\mathfrak{g})N_0) &\geq \dim(\mathcal{U}_k(\mathfrak{p}_l^+)N') \\ &= \dim(N') \dim(\mathcal{U}_k(\mathfrak{p}_l^+)) \quad (\text{part (e) of Lemma 2.6}) \\ &= \dim(N') \binom{k+n}{n} \geq \frac{\dim(N')}{n!} k^n. \end{aligned}$$

Comparing to (1), we conclude that  $\dim(N') \leq B$ , and the lemma follows.  $\square$



Define a partial order “ $\preceq$ ” on  $\Lambda$  by

$$\tau \preceq \tau' \quad \text{if } \tau' - \tau \in \Lambda^+.$$

Denote by  $W$  the Weyl group of the root system  $\Delta$ .

**Lemma 3.4.** Assume  $n \geq 2$ . If  $L_\lambda$  has Gelfand–Kirillov dimension  $n$ , then there is a  $\mu \in \Lambda_{\text{gnl}}$  such that

- (a)  $\mu \in W\lambda$ ,  $\mu \neq \lambda$ , and
- (b)  $\lambda \preceq \mu \preceq \lambda + m(\epsilon_1 + \epsilon_2)$  for some positive integer  $m$ .

**Proof.** By observing the composition series of the generalized Verma module  $M_\lambda$ , we write

$$\text{ch}(M_\lambda) = \text{ch}(L_\lambda) + c_1 \text{ch}(L_{\mu^1}) + c_2 \text{ch}(L_{\mu^2}) + \cdots + c_k \text{ch}(L_{\mu^k}), \quad (4)$$

where  $\mu^1, \mu^2, \dots, \mu^k$  are distinct elements of

$$(\Lambda_{\text{gnl}} \cap (W\lambda) \cap (\lambda + \Lambda^+)) \setminus \{\lambda\},$$

and  $c_1, c_2, \dots, c_k$  are positive integers. It is clear that

$$\text{ch}(M_\lambda) = \text{ch}(E_{\lambda+\rho} \otimes S(\mathfrak{p}_s^+)) \text{ch}(S(\mathfrak{p}_l^+)), \quad (5)$$

where “ $S$ ” stands for symmetric algebras. Since  $\mathcal{R}$  is an integral domain, (4), (5) and Lemma 3.2 implies

$$\text{ch}(E_{\lambda+\rho} \otimes S(\mathfrak{p}_s^+)) = \text{ch}(L_\lambda^{\mathfrak{p}_l^-}) + c_1 \text{ch}(L_{\mu^1}^{\mathfrak{p}_l^-}) + c_2 \text{ch}(L_{\mu^2}^{\mathfrak{p}_l^-}) + \cdots + c_k \text{ch}(L_{\mu^k}^{\mathfrak{p}_l^-}). \quad (6)$$

By Lemma 3.3,  $L_\lambda^{\mathfrak{p}_l^-}$  is finite-dimensional. Therefore  $\lambda + \rho + m(\epsilon_1 + \epsilon_2)$  is not a weight of it for some positive integer  $m$ . By (6), since  $\lambda + \rho + m(\epsilon_1 + \epsilon_2)$  is a weight of  $E_{\lambda+\rho} \otimes S(\mathfrak{p}_s^+)$ , it must be a weight of  $L_\mu$  for some  $\mu = \mu^1, \mu^2, \dots, \mu^k$ . This  $\mu$  fulfills the requirements of the lemma.  $\square$

**Proof of Proposition 3.1.** The simple roots of  $\Delta$  are

$$\alpha_1 = \epsilon_n - \epsilon_{n-1}, \quad \alpha_2 = \epsilon_{n-1} - \epsilon_{n-2}, \quad \dots, \quad \alpha_{n-1} = \epsilon_2 - \epsilon_1, \quad \alpha_n = 2\epsilon_1,$$

and the corresponding fundamental weights are

$$\varpi_1 = (0, 0, \dots, 0, 0, 1), \quad \varpi_2 = (0, 0, \dots, 0, 1, 1), \quad \dots, \quad \varpi_n = (1, 1, \dots, 1, 1, 1).$$

Let  $\mu = (\mu_1, \mu_2, \dots, \mu_k) \in \Lambda_{\text{gnl}}$  and  $m$  be as in Lemma 3.4. Condition (b) of Lemma 3.4 implies

$$\lambda \cdot \varpi_i \leq \mu \cdot \varpi_i \leq (\lambda + m(\epsilon_1 + \epsilon_2)) \cdot \varpi_i, \quad i = 1, 2, \dots, n. \quad (7)$$

Condition (a) of Lemma 3.4 implies

$$\{|\mu_1|, |\mu_2|, \dots, |\mu_n|\} = \{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\} \quad (8)$$

as sets counted with multiplicities. Now (7) and (8) force

$$\begin{cases} \text{(a)} & \mu_i = \lambda_i, \quad \text{for } i \geq 3, \\ \text{(b)} & \mu_2 \geq \lambda_2, \quad \mu_1 + \mu_2 \geq \lambda_1 + \lambda_2, \quad \text{and} \\ \text{(c)} & \{|\mu_1|, |\mu_2|\} = \{|\lambda_1|, |\lambda_2|\} \quad \text{as sets counted with multiplicities.} \end{cases} \quad (9)$$

Since  $\lambda, \mu \in \Lambda_{\text{gnl}}$ ,  $\mu \neq \lambda$ , and  $\lambda \preccurlyeq \mu$ , we have

$$\begin{cases} \text{(a)} & \mu_1, \mu_2, \lambda_1, \lambda_2 \in \frac{1}{2} + \mathbb{Z}, \\ \text{(b)} & \mu_1 > \mu_2, \quad \lambda_1 > \lambda_2, \\ \text{(c)} & \mu_1 + \mu_2 - \lambda_1 - \lambda_2 \in 2\mathbb{Z}, \quad \text{and} \\ \text{(d)} & (\mu_1, \mu_2) \neq (\lambda_1, \lambda_2). \end{cases} \quad (10)$$

Now it is elementary to conclude that  $-|\lambda_1| > \lambda_2$  from (10) and (b), (c) of (9).  $\square$

As a consequence of Proposition 3.1, we prove Corollary 1.2, i.e.,

**Corollary 3.5.** *Let  $\lambda' \in \Lambda_{\text{gnl}}$ . Assume both  $L_\lambda$  and  $L_{\lambda'}$  have Gelfand–Kirillov dimension  $n$ . If  $L_\lambda$  and  $L_{\lambda'}$  have the same infinitesimal character and the same central character, then  $\lambda = \lambda'$ .*

**Proof.** The case of  $n = 1$  is proved in Lemma 2.2. Assume  $n \geq 2$ , and  $L_\lambda$  and  $L_{\lambda'}$  have the same infinitesimal character. Then Proposition 3.1 implies

$$\lambda' = \lambda \quad \text{or} \quad (-\lambda_1, \lambda_2, \dots, \lambda_n).$$

Write  $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_n) \in \Lambda_{\text{gnl}}$ . Notice that  $L_\lambda$  and  $L_{\lambda'}$  have the same central character if and only if

$$(\lambda_1 + \lambda_2 + \dots + \lambda_n) - (\lambda'_1 + \lambda'_2 + \dots + \lambda'_n) \in 2\mathbb{Z}.$$

Therefore  $\lambda' = (-\lambda_1, \lambda_2, \dots, \lambda_n)$  implies that they have different central character. This finishes the proof.  $\square$

Now Corollary 1.3 can be proved as Lemma 2.4, i.e., we have

**Corollary 3.6.** *Every oscillator-like  $(\mathfrak{g}, K)$  module is completely reducible.*

#### 4. Tensor products of the oscillator representation with finite-dimensional ones

Now assume  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) \in \Lambda_{\text{osc}}$ . Write

$$\mu = (\mu_1, \mu_2, \dots, \mu_n) = \left( -|\lambda_1| + \frac{1}{2}, \lambda_2 + \frac{3}{2}, \dots, \lambda_n + \frac{2n-1}{2} \right) \in \mathbb{Z}^n.$$

Then  $0 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ . Let  $F_\mu$  be an irreducible finite-dimensional representation of  $G$  of lowest weight  $\mu$ .

For simplicity, write

$$\begin{aligned}\lambda_{\text{even}} &= \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots, -\frac{2n-1}{2}\right), \\ \lambda_{\text{odd}} &= \left(\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots, -\frac{2n-1}{2}\right), \\ \omega_{\text{even}} &= L_{\lambda_{\text{even}}},\end{aligned}$$

and

$$\omega_{\text{odd}} = L_{\lambda_{\text{odd}}}.$$

**Proposition 4.1.** *If  $\lambda_1 < 0$ , then  $L_\lambda$  is isomorphic to a submodule of  $\omega_{\text{even}} \otimes F_\mu$ . If  $\lambda_1 > 0$ , then  $L_\lambda$  is isomorphic to a submodule of  $\omega_{\text{odd}} \otimes F_\mu$ .*

Let  $u_0$  and  $v_0$  be non-zero lowest weight vectors of  $\omega_{\text{even}}$  and  $F_\mu$ , respectively. It is trivial that

**Lemma 4.2.** *The non-zero vector  $u_0 \otimes v_0$  is in  $(\omega_{\text{even}} \otimes F_\mu)^{n^-}$ . If  $\lambda_1 < 0$ , then it has weight  $\lambda + \rho$ .*

Let  $\{H_1, X_1, Y_1\} \subset \mathfrak{g}$  be a triple such that

- (a)  $X_1$  is a root vector corresponding to the root  $2\epsilon_1$ ,
- (b)  $Y_1$  is a root vector corresponding to the root  $-2\epsilon_1$ , and
- (c)  $[X_1, Y_1] = H_1$ ,  $[H_1, X_1] = 2X_1$ ,  $[H_1, Y_1] = -2Y_1$ .

Define a sequence  $x_0, x_1, x_2, \dots$  of non-zero elements in  $\omega_{\text{odd}}$  by

- (a)  $x_0$  is a lowest weight vector, and
- (b)  $x_i = \frac{2}{2i^2 + i} X_1 x_{i-1}$ ,  $i = 1, 2, \dots$

We calculate that (as in [12, Lemma 1.2.4])

$$Y_1(x_i) = -x_{i-1}, \quad i = 1, 2, \dots$$

Define a sequence  $y_0, y_1, \dots, y_{|\mu_1|}$  of non-zero elements in  $F_\mu$  by

- (a)  $y_0$  is a weight vector of extremal weight  $(|\mu_1|, \mu_2, \dots, \mu_n)$ , and
- (b)  $y_i = Y_1 y_{i-1}$ ,  $i = 1, 2, \dots, |\mu_1|$ .

**Lemma 4.3.** *The non-zero vector  $\sum_{i=0}^{|\mu_1|} x_i \otimes y_i$  is in  $(\omega_{\text{odd}} \otimes F_\mu)^{n^-}$ . If  $\lambda_1 > 0$ , then it has weight  $\lambda + \rho$ .*

**Proof.** It is easy to see that  $\sum_{i=0}^{|\mu_1|} x_i \otimes y_i$  has weight  $\lambda_{\text{odd}} + \rho + \mu + 2|\mu_1|\epsilon_1$ , which is  $\lambda + \rho$  when  $\lambda_1 > 0$ . Every weight of  $\omega_{\text{odd}} \otimes F_\mu$  belongs to  $\lambda_{\text{odd}} + \rho + \mu + \Lambda^+$ . Recall that  $2\epsilon_1$  is a simple root. For any other simple root  $\alpha$ , we claim that

$$-\alpha + (\lambda_{\text{odd}} + \rho + \mu + 2|\mu_1|\epsilon_1) \notin \lambda_{\text{odd}} + \rho + \mu + \Lambda^+,$$

or equivalently,

$$-\alpha + 2|\mu_1|\epsilon_1 \notin \Lambda^+.$$

Otherwise we get a contradiction by pairing the above with the fundamental weight  $\varpi_\alpha$  corresponding to  $\alpha$ .

Let  $Y_{-\alpha}$  be a root vector corresponding to  $-\alpha$ . Now  $Y_{-\alpha} \sum_{i=0}^{|\mu_1|} x_i \otimes y_i = 0$  since it has weight  $-\alpha + (\lambda_{\text{odd}} + \rho + \mu + 2|\mu_1|\epsilon_1)$ , which is not a weight of  $\omega_{\text{odd}} \otimes F_\mu$ .

On the other hand

$$Y_1 \left( \sum_{i=0}^{|\mu_1|} x_i \otimes y_i \right) = \sum_{i=1}^{|\mu_1|} -x_{i-1} \otimes y_i + \sum_{i=0}^{|\mu_1|-1} x_i \otimes y_{i+1} = 0.$$

The lemma now follows from the fact that the root vectors

$$\{Y_{-\alpha} \mid \alpha \text{ is a simple root}\}$$

generate the Lie algebra  $\mathfrak{n}^-$ .  $\square$

When  $\lambda_1 < 0$ , Lemma 4.2 implies that  $L_\lambda$  is a subquotient of  $\omega_{\text{even}} \otimes F_\mu$ , and is therefore a submodule by Corollary 1.3. This proves the first assertion of Proposition 4.1. Similarly, the second assertion is implied by Lemma 4.3.

In conclusion, by Proposition 3.1, Proposition 4.1, and the fact that tensoring with a finite-dimensional representation does not increase the Gelfand–Kirillov dimension, we have that for every  $\lambda \in \Lambda_{\text{gnl}}$ , the following three conditions are equivalent.

- (a)  $L_\lambda$  has Gelfand–Kirillov dimension  $n$ .
- (b)  $\lambda \in \Lambda_{\text{osc}}$ .
- (c)  $L_\lambda$  occurs in  $\omega \otimes F$  for some finite-dimensional representation  $F$ .

This implies both Theorem 1.1 and Corollary 1.4.

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